ON THE ACTION OF THE DUAL GROUP ON THE COHOMOLOGY OF PERVERSE SHEAVES ON THE AFFINE GRASSMANNIAN

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Abstract. It was proved by Ginzburg and Mirkovic-Vilonen that the $G(O)$-equivariant perverse sheaves on the affine grassmannian of a connected reductive group G form a tensor category equivalent to the tensor category of finite dimensional representations of the dual group $G^\vee$. The proof use the Tannakian formalism. The purpose of this paper is to construct explicitely the action of $G^\vee$ on the global cohomology of a perverse sheaf. It would be interesting to find a q-analogue of this construction. It would give the global counterpart to [BG].

1. Notations and reminder on affine Grassmanians

1.1. Let G be a connected reductive complex algebraic group. Let B, T, be a Borel and a Cartan subgroup of G. Let $U \subset B$ be the unipotent radical of B. Let B^- a Borel subgroup such that $B \cap B^- = T$. Set $X_T = \mathrm{Hom}\,(T, \mathbb{G}_m)$ and $X_T^\vee = \mathrm{Hom}\,(\mathbb{G}_m, T)$ be the weight an the coweight lattice of G. For simplicity we write $X = X_T$ and $X^\vee = X_T^\vee$. Let $(\ ,\ ) : X \times X^\vee \to \mathbb{Z}$ be the natural pairing. Let R be the set of roots, $R^\vee$ the set of coroots. Let $R_\pm \subset R$, $R_\pm^\vee \subset R^\vee$, be the subsets of positive and negative roots and coroots. Let $X_+ \subset X$, $X_+^\vee \subset X^\vee$, be the subsets of dominant weights and coweights. Let $\rho_G \in X$ be half the sum of all positive roots. If there is no ambiguity we simply write ρ instead of ρ_G. Let $G^\vee$ and $Z(G)$ be the dual group and the center of G. Let $\alpha_i, \alpha_i^\vee$, $i \in I$, be the simple roots and the simple coroots, and let $\omega_i, \omega_i^\vee$ be the fundamental weights and coweights. For any root $\alpha \in R$ let $U_\alpha \subset G$ be the corresponding root subgroup. If $\alpha = \alpha_i$, $i \in I$, we simply set $U_i = U_{\alpha_i}$ and $U_i^- = U_{-\alpha_i}$. Let W be the Weyl group of G. For any $i \in I$ let s_i be the simple reflexion corresponding to the simple root α_i.

1.2. Let $K = \mathbb{C}((t))$ be the field of Laurent formal series, and let $O = \mathbb{C}[[t]]$ be the subring of integers. Recall that $G(O)$ is a group scheme and that $G(K)$ is a group ind-scheme. The quotient set $\mathrm{Gr}^G = G(K)/G(O)$ is endowed with the structure of an ind-scheme. We may write Gr instead of Gr^G, hoping that it makes no confusion. For any coweight $\lambda^\vee \in X^\vee$, let $t^{\lambda^\vee} \in T(K)$ be the image of t by the group homomorphism $\lambda^\vee : \mathbb{G}_m(K) \to T(K)$. If $\lambda^\vee$ is dominant, set $e_{\lambda^\vee} = t^{\lambda^\vee} G(O)/G(O) \in \mathrm{Gr}$. The $G(O)$-orbit $\mathrm{Gr}_{\lambda^\vee} = G(O) \cdot e_{\lambda^\vee}$ is connected and simply connected. Let $\overline{\mathrm{Gr}}_{\lambda^\vee}$ be its Zariski closure. Let $\mathcal{P}_G$ be the category of

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$G(O)$-equivariant perverse sheaves on Gr. For any $\lambda^\vee$ let $\mathcal{IC}_{\lambda^\vee}$ be the intersection cohomology complex on $\mathrm{Gr}_{\lambda^\vee}$ with coefficients in $\mathbb{C}$. Consider the fiber product $G(K)\times_{G(O)}\mathrm{Gr}$. It is the quotient of $G(K)\times\mathrm{Gr}$ by $G(O)$, where $u\in G(O)$ acts on $G(K)\times\mathrm{Gr}$ by $(g,x)\mapsto(gu^{-1},ux)$. The map

$$\tilde p\,:\,G(K)\times_{G(O)}\mathrm{Gr}\to\mathrm{Gr},\quad (g,x)\mapsto ge_0$$

is the locally trivial fibration with fiber Gr associated to the $G(O)$-bundle

$$p\,:\,G(K)\to\mathrm{Gr}.$$

Thus $G(K)\times_{G(O)}\mathrm{Gr}$ is an ind-scheme : it is the inductive limit of the subschemes $p^{-1}(\overline{\mathrm{Gr}}_{\lambda_1^\vee})\times_{G(O)}\overline{\mathrm{Gr}}_{\lambda_2^\vee}$. Consider also the map

$$m\,:\,G(K)\times_{G(O)}\mathrm{Gr}\to\mathrm{Gr},\quad (g,x)\mapsto gx.$$

For any $\lambda_1^\vee,\lambda_2^\vee\in X_+^\vee$ let $\mathcal{IC}_{\lambda_1^\vee}\star\mathcal{IC}_{\lambda_2^\vee}$ be the direct image by m of the intersection cohomology complex of the subvariety

$$p^{-1}(\overline{\mathrm{Gr}}_{\lambda_1^\vee})\times_{G(O)}\overline{\mathrm{Gr}}_{\lambda_2^\vee}\subset G(K)\times_{G(O)}\mathrm{Gr}.$$

The complex $\mathcal{IC}_{\lambda_1^\vee}\star\mathcal{IC}_{\lambda_2^\vee}$ is perverse (see [MV], and [NP, Corollaire 9.7] for more details). It is known that the cohomology sheaves of the complex $\mathcal{IC}_{\lambda^\vee}$ are pure by the argument similar to [KT] (see also [KL]). It is also known that any object in $\mathcal{P}_G$ is a direct sum of complexes $\mathcal{IC}_{\lambda^\vee}$ (see [BD, Proposition 5.3.3.(i)] for a proof). Thus we get a convolution product $\star\,:\,\mathcal{P}_G\times\mathcal{P}_G\to\mathcal{P}_G$. It is the convolution product defined by Mirkovic and Vilonen.

1.3. Let $P\subset G$ be a parabolic subgroup of G, $N\subset P$ be the unipotent radical, $M=P/N$ be the Levi factor. Let $M'=[M,M]$ be the semisimple part of M. Consider the diagram

$$\mathrm{Gr}^G\xleftarrow{\gamma}\mathrm{Gr}^P\xrightarrow{\pi}\mathrm{Gr}^M,$$

where the maps γ and π are induced by the embedding $P\subset G$ and the projection $P\to M$. The fibers of π are $N(K)$-orbits. Observe that Gr^M is not connected. The connected components of Gr^M are labelled by characters of the center of the dual group $M^\vee$. Let $\mathrm{Gr}^{M,\theta^\vee}\subset\mathrm{Gr}^M$ be the component associated to $\theta^\vee\in X_{Z(M^\vee)}$. By definition, $e_{\lambda^\vee}\in\mathrm{Gr}^{M,\theta^\vee}$ if and only if the restriction of $\lambda^\vee$ to $Z(M^\vee)$ coincides with $\theta^\vee$. The element $\rho-\rho_M$ belongs to $X^\vee_{Z(M^\vee)}$. Put

$$\mathrm{Gr}^{M,n}=\bigsqcup_{2(\theta,\rho-\rho_M)=n}\mathrm{Gr}^{M,\theta^\vee}.$$

The following facts are proved in [BD, Section 5.3].

Proposition. *(a) The functor $\pi_!\gamma^*$ gives a map $res^{GM}\,:\,\mathcal{P}_G\to\tilde{\mathcal{P}}_M=\bigoplus_n\mathcal{P}_{M,n}[-n]$, where $\mathcal{P}_{M,n}$ is the subcategory of $M(O)$-equivariant perverse sheaves on $Gr^{M,n}$.*
(b) For any $\mathcal{E},\mathcal{F}\in\mathcal{P}_G$ we have $res^{GM}(\mathcal{E}\star\mathcal{F})=(res^{GM}\mathcal{E})\star(res^{GM}\mathcal{F})$.
(c) For any $\mathcal{E}\in\mathcal{P}_G$ we have $H^*(Gr,\mathcal{E})=H^*(Gr^M,res^{GM}\mathcal{E})$.
(d) If $P_1\subset P$ is a parabolic subgroup and M_1 is its Levi factor then res^{MM_1} maps $\tilde{\mathcal{P}}_M$ to $\tilde{\mathcal{P}}_{M_1}$, and $res^{GM_1}=res^{MM_1}\circ res^{GM}$. □

1.4. Let $\tilde{\mathfrak{g}}$ be the affine Kac-Moody Lie algebra associated to G. Let $\tilde{\omega}_0$ be the fundamental weight of $\tilde{\mathfrak{g}}$ which is trivial on $\mathrm{Lie}(T)$. Let W_0 be the irreducible integrable highest weight module of $\tilde{\mathfrak{g}}$ with higest weight $\tilde{\omega}_0$. Let π be the corresponding group homomorphism $G(K) \to PGL(W_0)$ (see [Ku, Appendix C] for instance). The central extension $\tilde{G}(K)$ of $G(K)$ is the pull-back $\pi^* GL(W_0)$, where $GL(W_0)$ must be viewed as a $\mathbb{C}^\times$-principal bundle on $PGL(W_0)$. The restriction of the central extension to $G(O)$, denoted by $\tilde{G}(O)$, splits, i.e. $\tilde{G}(O) = G(O) \times \mathbb{C}^\times$. Fix a highest weight vector $w_0 \in W_0$. Let $\mathcal{L}_G$ be the pull-back of $O_{\mathbb{P}}(1)$ by the embedding of ind-schemes $\iota : \mathrm{Gr}^G \hookrightarrow \mathbb{P}(W_0)$ induced by the map

$$G(K) \to \mathbb{P}(W_0),\ g \mapsto [\mathbb{C} \cdot g w_0].$$

The sheaf $\mathcal{L}_G$ is obviously algebraic.

1.5. For any $i \in I$ let P_i be the corresponding subminimal parabolic subgroup of G. Let $N_i \subset P_i$ be the unipotent radical and put $M_i = P_i/N_i$. Hereafter we set $i\mathrm{res} = \mathrm{res}^{GM_i}$, $\pi_i = \pi$, $\gamma_i = \gamma$, $Z_i = Z(M_i)$ and $\mathcal{L}_i = \mathcal{L}_{M_i}$. The product by the first Chern class of $\mathcal{L}_i$ gives a map

$$l_i : H^*(\mathrm{Gr}^{M_i}, \mathcal{E}) \to H^{*+2}(\mathrm{Gr}^{M_i}, \mathcal{E}),$$

for any $\mathcal{E} \in \mathcal{P}_{M_i}$.

1.6. For any $\mu^\vee \in X^\vee$ set $S_{\mu^\vee} = U(K) \cdot e_{\mu^\vee}$. It was proved by Mirkovic and Vilonen that if $\mathcal{E} \in \mathcal{P}_G$ then

$$(a) \qquad H^*(\mathrm{Gr}, \mathcal{E}) = \bigoplus_{\mu^\vee \in X^\vee} H_c^{2(\rho,\mu^\vee)}(S_{\mu^\vee}, \mathcal{E}),$$

(see [MV], and [NP] for more details). For any $i \in I$ and any $\mu^\vee \in X^\vee$ set also $S^{M_i}_{\mu^\vee} = U_i(K) \cdot e_{\mu^\vee} \subset \mathrm{Gr}^{M_i}$. The grassmanian Gr^{M_i} may be viewed as the set of points of Gr which are fixed by the action of the group Z_i by left translations. This fixpoints subset is denoted by ${}^{Z_i}\mathrm{Gr}$. In particular, $S^{M_i}_{\mu^\vee}$ may be viewed as a subset of Gr.

2. Construction of the operators $\mathbf{e}_i$, $\mathbf{f}_i$, $\mathbf{h}_i$

2.1. To avoid useless complications, hereafter we assume that G is semi-simple. The generalization to the reductive case is immediate. For any $i \in I$ and $\mathcal{E} \in \mathcal{P}_G$, let $\mathbf{e}_i$ be the composition of the chain of maps

$$H^*(\mathrm{Gr}, \mathcal{E}) = H^*(\mathrm{Gr}^{M_i}, i\mathrm{res}\,\mathcal{E}) \xrightarrow{l_i} H^{*+2}(\mathrm{Gr}^{M_i}, i\mathrm{res}\,\mathcal{E}) = H^{*+2}(\mathrm{Gr}, \mathcal{E}).$$

Moreover, set

$$\mathbf{h}_i = \bigoplus_{\lambda^\vee \in X^\vee} (\alpha_i, \lambda^\vee)\, id_{H_c^*(S_{\lambda^\vee}, \mathcal{E})} : H^*(\mathrm{Gr}, \mathcal{E}) \to H^*(\mathrm{Gr}, \mathcal{E}).$$

By the hard Lefschetz theorem there is a unique linear operator $\mathbf{f}_i : H^*(\mathrm{Gr}, \mathcal{E}) \to H^{*-2}(\mathrm{Gr}, \mathcal{E})$ such that $(\mathbf{e}_i, \mathbf{h}_i, \mathbf{f}_i)$ is a $\mathfrak{sl}(2)$-triple.

Theorem. *For any $\mathcal{E} \in \mathcal{P}_G$, the operators $\mathbf{e}_i, \mathbf{f}_i, \mathbf{h}_i$, with $i \in I$, give an action of the dual group $G^\vee$ on the cohomology $H^*(Gr, \mathcal{E})$.* $\square$

2.2. The rest of the paper is devoted to the proof of the theorem.

Lemma. *For all* $\lambda^\vee \in X^\vee$ *we have*

$$\mathbf{e}_i(H_c^*(S_{\lambda^\vee},\mathcal{E})) \subset H_c^*(S_{\lambda^\vee+\alpha_i^\vee},\mathcal{E}) \quad \textit{and} \quad \mathbf{f}_i(H_c^*(S_{\lambda^\vee},\mathcal{E})) \subset H_c^*(S_{\lambda^\vee-\alpha_i^\vee},\mathcal{E}).$$

Proof. It is sufficient to check the first claim. Since

$$S_{\lambda^\vee} = N_i(K)U_i(K)\cdot e_{\lambda^\vee} = \pi_i^{-1}(S_{\lambda^\vee}^{M_i}),$$

we get, for any $\mathcal{E} \in \mathcal{P}_G$,

$$(a) \qquad H_c^*(S_{\lambda^\vee},\mathcal{E}) = H_c^*(S_{\lambda^\vee}^{M_i}, i\mathrm{res}\,\mathcal{E}).$$

Now, if $\mathcal{E} \in \mathcal{P}_{M_i}$ then

$$l_i(H_c^*(S_{\lambda^\vee}^{M_i},\mathcal{E})) = l_i(H_c^{(\alpha_i,\lambda^\vee)}(S_{\lambda^\vee}^{M_i},\mathcal{E})) \subset H_c^{2+(\alpha_i,\lambda^\vee)}(\mathrm{Gr}^{M_i},\mathcal{E}).$$

Moreover, for all $\mu^\vee \in X^\vee \simeq X_{T^\vee}$ we have

$$S_{\mu^\vee} \cap \mathrm{Gr}^{M_i,\theta^\vee} \neq \emptyset \iff \mu^\vee|_{Z(M_i^\vee)} = \theta^\vee.$$

Thus, if $\mathcal{E} \in \mathcal{P}_{M_i}$ then

$$l_i(H_c^*(S_{\lambda^\vee}^{M_i},\mathcal{E})) = \bigoplus_{\mu^\vee} H_c^{(\alpha_i,\mu^\vee)}(S_{\mu^\vee}^{M_i},\mathcal{E}),$$

where the sum is over all $\mu^\vee \in X^\vee \simeq X_{T^\vee}$ such that

$$\mu^\vee|_{Z(M_i^\vee)} = \lambda^\vee|_{Z(M_i^\vee)} \quad \text{and} \quad (\alpha_i,\mu^\vee) = (\alpha_i,\lambda^\vee+\alpha_i^\vee).$$

The only possibility is $\mu^\vee = \lambda^\vee + \alpha_i^\vee$. □

The lemma implies that $[\mathbf{h}_i,\mathbf{e}_j] = (\alpha_i,\alpha_j^\vee)\,\mathbf{e}_j$ for all $i,j\in I$. Since $\mathbf{e}_i,\mathbf{f}_i$, are locally nilpotent and since $[\mathbf{e}_i,\mathbf{f}_i] = \mathbf{h}_i$ by construction, if $[\mathbf{e}_i,\mathbf{f}_j]=0$ for any $i\neq j$ then the operators $\mathbf{e}_i,\mathbf{f}_i,\mathbf{h}_i$, give a representation of the Lie algebra $\mathfrak{g}^\vee$ of $G^\vee$ on the cohomology group $H^*(\mathrm{Gr},\mathcal{E})$ for any $\mathcal{E}\in\mathcal{P}_G$ (see [Ka, Section 3.3]). The action of the operators $\mathbf{h}_i$ lifts to an action of the torus of $G^\vee$. Thus, the representation of the Lie algebra $\mathfrak{g}^\vee$ lifts to a representation of the group $G^\vee$. By (1.3.d), in order to check the relation $[\mathbf{e}_i,\mathbf{f}_j]=0$ for $i\neq j$ we can assume that the group G has rank 2.

2.3. Recall that any complex $\mathcal{IC}_{\lambda^\vee}$ is a direct factor of a product $\mathcal{IC}_{\lambda_1^\vee}\star\mathcal{IC}_{\lambda_2^\vee}\star\cdots\star\mathcal{IC}_{\lambda_n^\vee}$ such that the coweights $\lambda_i^\vee$ are either minuscule or quasi-minuscule (see [NP, Proposition 9.6]). Observe that [NP, Lemmes 10.2, 10.3] imply indeed that if the set of minuscule coweights is non empty, then we can find such a product with all the $\lambda_i^\vee$'s beeing minuscule. Recall also that for any $\mathcal{E},\mathcal{F}\in\mathcal{P}_G$ there is a canonical isomorphism of graded vector spaces

$$(a) \qquad H_c^*(S_{\lambda^\vee},\mathcal{E}\star\mathcal{F}) \simeq \bigoplus_{\mu^\vee+\nu^\vee=\lambda^\vee} H_c^*(S_{\mu^\vee},\mathcal{E})\otimes H_c^*(S_{\nu^\vee},\mathcal{F}),$$

(see [MV], and [NP, Proof of Theorem 3.1] for more details). Let $\Delta(\mathbf{e}_i), \Delta(\mathbf{f}_i), \Delta(\mathbf{h}_i)$, be the composition

$$H^*(\mathrm{Gr}, \mathcal{E}) \otimes H^*(\mathrm{Gr}, \mathcal{F}) = H^*(\mathrm{Gr}, \mathcal{E} \star \mathcal{F}) \xrightarrow{\mathbf{e}_i, \mathbf{f}_i, \mathbf{h}_i} H^*(\mathrm{Gr}, \mathcal{E} \star \mathcal{F}) = H^*(\mathrm{Gr}, \mathcal{E}) \otimes H^*(\mathrm{Gr}, \mathcal{F}),$$

where the equalities are given by (1.6.a) and (a).

Lemma. *If $x = \mathbf{e}_i, \mathbf{f}_i, \mathbf{h}_i$, then $\Delta(x) = x \otimes 1 + 1 \otimes x$.*

Proof. If $x = \mathbf{h}_i$ the equation is obvious. If $x = \mathbf{f}_i$ it is a direct consequence of the two others since a $\mathfrak{sl}(2)$-triple $(\mathbf{e}_i, \mathbf{h}_i, \mathbf{f}_i)$ is completely determined by $\mathbf{e}_i$ and $\mathbf{h}_i$. Thus, from (2.3.a), (1.3.b) and (1.3.c), it suffices to check the equality when $G = \mathrm{SL}(2)$ and $x = \mathbf{e}_i$. Then, the operator $\mathbf{e}_i$ is the product by the 1-st Chern class of the line bundle $\mathcal{L}_{\mathrm{SL}(2)}$ on $\mathrm{Gr}^{\mathrm{SL}(2)}$. More generally, for any simply connected group G, the $G(O)$-equivariant line bundle $\mathcal{L}_G$ on the grassmannian Gr lifts uniquely to a $G(K)$-equivariant line bundle on the ind-scheme $G(K) \times_{G(O)} \mathrm{Gr}$. Let denote it by $\mathcal{L}_2$. The group $G(O)$ acts on the pull-back of $\mathcal{L}_G$ by the projection $G(K) \times \mathrm{Gr} \to \mathrm{Gr}$. The quotient is the bundle $\mathcal{L}_2$. The vector bundle $\mathcal{L}_2$ is algebraic, i.e. its restriction to the subscheme $p^{-1}(\overline{\mathrm{Gr}}_{\lambda_1^\vee}) \times_{G(O)} \overline{\mathrm{Gr}}_{\lambda_2^\vee}$ is an algebraic vector bundle for any $\lambda_1^\vee$, $\lambda_2^\vee$. Indeed, there is a normal pro-unipotent closed subgroup H of $G(O)$ such that $G(O)/H$ is finite dimensional and H acts trivialy on $\overline{\mathrm{Gr}}_{\lambda_1^\vee}, \overline{\mathrm{Gr}}_{\lambda_2^\vee}$. Since H is pro-unipotent, the restriction of $\mathcal{L}_G$ to $\overline{\mathrm{Gr}}_{\lambda_2^\vee}$ is $G(O)/H$-equivariant. Thus the restriction of $\mathcal{L}_2$ to $p^{-1}(\overline{\mathrm{Gr}}_{\lambda_1^\vee}) \times_{G(O)} \overline{\mathrm{Gr}}_{\lambda_2^\vee}$ is identified with the algebraic sheaf on

$$(p^{-1}(\overline{\mathrm{Gr}}_{\lambda_1^\vee})/H) \times_{G(O)/H} \overline{\mathrm{Gr}}_{\lambda_2^\vee}$$

induced by the restriction of $\mathcal{L}_G$ to $\overline{\mathrm{Gr}}_{\lambda_2^\vee}$. Consider also the pull-back $\mathcal{L}_1$ of the line bundle $\mathcal{L}_G$ by the 1-st projection $\tilde{p} : G(K) \times_{G(O)} \mathrm{Gr} \to \mathrm{Gr}$. We claim that

$$(b) \qquad m^* \mathcal{L}_G = \mathcal{L}_1 \otimes \mathcal{L}_2.$$

Let $\mu : G(K) \times G(K) \to G(K)$ be the multiplication map. The product in the group $\tilde{G}(K)$ gives an isomorphism of bundles

$$\mu^* p^* \mathcal{L}_G \simeq p^* \mathcal{L}_G \boxtimes p^* \mathcal{L}_G$$

on $G(K) \times G(K)$. This isomorphism descends to the fiber product $G(K) \times_{G(O)} \mathrm{Gr}$ and implies (b). Observe now that (a) is induced by the canonical isomorphism

$$\left(p^{-1}(\overline{\mathrm{Gr}}_{\lambda_1^\vee}) \times_{G(O)} \overline{\mathrm{Gr}}_{\lambda_2^\vee}\right) \cap m^{-1}(S_{\lambda^\vee}) \simeq \bigsqcup_{\mu^\vee + \nu^\vee = \lambda^\vee} (S_{\mu^\vee} \cap \overline{\mathrm{Gr}}_{\lambda_1^\vee}) \times (S_{\nu^\vee} \cap \overline{\mathrm{Gr}}_{\lambda_2^\vee})$$

resulting from the local triviality of p. Let $l_{\mathcal{E}}$ be the product by the 1-st Chern class of $\mathcal{L}_G$ on the global cohomology of the perverse sheaf $\mathcal{E} \in \mathcal{P}_G$. Then (a) and (b) give $l_{\mathcal{E} \star \mathcal{F}} = l_{\mathcal{E}} \otimes 1 + 1 \otimes l_{\mathcal{F}}$. □

2.4. From Section 2.3 and (1.3.d) we are reduced to check the relation $[\mathbf{e}_i, \mathbf{f}_j] = 0$, $i \neq j$, on the cohomology group $H^*(\mathrm{Gr}, \mathcal{E})$ when G has rank 2 and $\mathcal{E} = \mathcal{IC}_{\lambda^\vee}$, with $\lambda^\vee$ minuscule or quasi-minuscule. For any dominant coroot $\lambda^\vee$ let $\Omega(\lambda^\vee) \subset X^\vee$ be the set of weights of the simple $G^\vee$-module with highest weight $\lambda^\vee$. Recall that

(a) the coweight $\lambda^\vee \in X_+^\vee - \{0\}$ is minuscule if and only if $\Omega(\lambda^\vee) = W \cdot \lambda^\vee$, if and only if $(\alpha, \lambda^\vee) = 0, \pm 1$, for all $\alpha \in R$,
(b) the coweight $\lambda^\vee \in X_+^\vee - \{0\}$ is quasi-minuscule if and only if $\Omega(\lambda^\vee) = W \cdot \lambda^\vee \cup \{0\}$, if and only if $\lambda^\vee$ is a maximal short coroot. Moreover if $\lambda^\vee$ is quasi-minuscule then $(\alpha, \lambda^\vee) = 0, \pm 1$, for all $\alpha \in R - \{\pm\lambda\}$.

For any coweight $\lambda^\vee$ we consider the isotropy subgroup $G_{\lambda^\vee}$ of $e_{\lambda^\vee}$ in G. Thus

$$G_{\lambda^\vee} = T \prod_{(\alpha, \lambda^\vee) \leq 0} U_\alpha.$$

In particular $B^- \subset G_{\lambda^\vee}$ if $\lambda^\vee$ is dominant, and we can consider the line bundle $\mathcal{L}(\lambda)$ on $G/G_{\lambda^\vee}$ associated to the weight λ. The structure of $\overline{\mathrm{Gr}}_{\lambda^\vee}$ for $\lambda^\vee$ minuscule or quasi-minuscule is described as follows in [NP].

Proposition. *(c) If $S_{\mu^\vee} \cap \overline{Gr}_{\lambda^\vee} \neq \emptyset$, then $\mu^\vee \in \Omega(\lambda^\vee)$.*
(d) If $\mu^\vee \in W \cdot \lambda^\vee$, then $S_{\mu^\vee} \cap \overline{Gr}_{\lambda^\vee} = S_{\mu^\vee} \cap Gr_{\lambda^\vee}$.
(e) If $\lambda^\vee \in \mathrm{X}_+^\vee$ is minuscule, then

$$\overline{Gr}_{\lambda^\vee} = Gr_{\lambda^\vee} = G/G_{\lambda^\vee} \quad \text{and} \quad S_{w\cdot\lambda^\vee} \cap Gr_{\lambda^\vee} \simeq UwG_{\lambda^\vee}/G_{\lambda^\vee} \quad \forall w \in W.$$

(f) Assume that $\lambda^\vee \in X_+^\vee$ is quasi-minuscule. Then $Gr_{\lambda^\vee} \simeq \mathcal{L}(\lambda)$ and $\overline{Gr}_{\lambda^\vee} \simeq \mathcal{L}(\lambda) \cup \{e_0\}$ as a G-varieties. Moreover,

$$S_{w\cdot\lambda^\vee} \cap Gr_{\lambda^\vee} \simeq \begin{cases} UwG_{\lambda^\vee}/G_{\lambda^\vee} & \text{if} \quad w \cdot \lambda \in R_-, \\ \\ \mathcal{L}|_{UwG_{\lambda^\vee}/G_{\lambda^\vee}} & \text{if} \quad w \cdot \lambda \in R_+. \end{cases}$$

□

3. Proof of the relation $[\mathbf{e}_i, \mathbf{f}_j] = 0$

3.1. Assume that G has rank two and set $I = \{1, 2\}$. The Bruhat decomposition for M_i implies that $M_i e_{\lambda^\vee} = U_i e_{s_i\cdot\lambda^\vee} \cup U_i U_{-\alpha_i^\vee} e_{\lambda^\vee}$. Thus,
(a) if $(\alpha_i, \lambda^\vee) > 0$ then $M_i e_{\lambda^\vee} = M_i e_{s_i\cdot\lambda^\vee} = U_i e_{\lambda^\vee} \cup \{e_{s_i\cdot\lambda^\vee}\}$,
(b) if $(\alpha_i, \lambda^\vee) = 0$ then $M_i e_{\lambda^\vee} = \{e_{\lambda^\vee}\}$.

3.2. Assume that $\lambda^\vee$ is a minuscule dominant coweight. Fix $\mu^\vee = w \cdot \lambda^\vee$ with $w \in W$, and fix $i \in I$. One of the following three cases holds
(a) we have $(\alpha_i, \mu^\vee) = 1$, and

$$S_{\mu^\vee}^{M_i} \cap \mathrm{Gr}_{\lambda^\vee} = U_i e_{\mu^\vee}, \quad S_{\mu^\vee - \alpha_i^\vee}^{M_i} \cap \mathrm{Gr}_{\lambda^\vee} = \{e_{\mu^\vee - \alpha_i^\vee}\}, \quad \mathrm{Gr}_{\mu^\vee}^{M_i} = U_i e_{\mu^\vee} \cup \{e_{\mu^\vee - \alpha_i^\vee}\},$$

(b) we have $(\alpha_i, \mu^\vee) = -1$, and

$$S_{\mu^\vee + \alpha_i^\vee}^{M_i} \cap \mathrm{Gr}_{\lambda^\vee} = U_i e_{\mu^\vee + \alpha_i^\vee}, \quad S_{\mu^\vee}^{M_i} \cap \mathrm{Gr}_{\lambda^\vee} = \{e_{\mu^\vee}\}, \quad \mathrm{Gr}_{\mu^\vee}^{M_i} = U_i e_{\mu^\vee + \alpha_i^\vee} \cup \{e_{\mu^\vee}\},$$

(c) we have $(\alpha_i, \mu^\vee) = 0$, and

$$S_{\mu^\vee}^{M_i} \cap \mathrm{Gr}_{\lambda^\vee} = \mathrm{Gr}_{\mu^\vee}^{M_i} = \{e_{\mu^\vee}\}.$$

Obviously, the sheaf $i\mathrm{res}\,\mathcal{IC}_{\lambda^\vee}$ is supported on $\mathrm{Gr}_{\lambda^\vee} \cap \mathrm{Gr}^{M_i}$. Thus, by (2.2.a) and Lemma 2.2, if $\mathbf{f}_2\mathbf{e}_1(H_c^*(S_{\mu^\vee}, \mathcal{IC}_{\lambda^\vee})) \neq \{0\}$ then

$$S^{M_1}_{\mu^\vee} \cap \mathrm{Gr}_{\lambda^\vee}, \quad S^{M_1}_{\mu^\vee+\alpha_1^\vee} \cap \mathrm{Gr}_{\lambda^\vee}, \quad S^{M_2}_{\mu^\vee+\alpha_1^\vee} \cap \mathrm{Gr}_{\lambda^\vee}, \quad S^{M_2}_{\mu^\vee+\alpha_1^\vee-\alpha_2^\vee} \cap \mathrm{Gr}_{\lambda^\vee}$$

are non empty. In particular, we get

$$(\alpha_1, \mu^\vee) = -1, \qquad (\alpha_2, \mu^\vee + \alpha_1^\vee) = 1.$$

Since $\lambda^\vee$ is minuscule, $\mu^\vee \in W \cdot \lambda^\vee$, and $(\alpha_2, \alpha_1^\vee) \leq 0$, we get

$$(\alpha_2, \mu^\vee) = 1, \quad (\alpha_1, \mu^\vee) = -1, \quad (\alpha_2, \alpha_1^\vee) = 0.$$

Similarly,

$$\mathbf{e}_1\mathbf{f}_2(H_c^*(S_{\mu^\vee}, \mathcal{IC}_{\lambda^\vee})) \neq \{0\} \Rightarrow (\alpha_2, \mu^\vee) = 1, \quad (\alpha_1, \mu^\vee) = -1, \quad (\alpha_1, \alpha_2^\vee) = 0.$$

Thus we are reduced to the case where $G = \mathrm{SL}(2) \times \mathrm{SL}(2)$, $M_1 \simeq \mathrm{SL}(2) \times \{1\}$, $M_2 = \{1\} \times \mathrm{SL}(2)$, $\lambda^\vee = \omega_1^\vee + \omega_2^\vee$, $\mu^\vee = -\omega_1^\vee + \omega_2^\vee$, and $\mathcal{IC}_{\lambda^\vee}$ is the constant sheaf on $\mathrm{Gr}_{\lambda^\vee}$. Then,

$$\mathrm{Gr}_{\lambda^\vee} \simeq \mathbb{P}^1 \times \mathbb{P}^1, \quad \mathrm{Gr}^{M_1} \cap \mathrm{Gr}_{\lambda^\vee} \simeq \mathbb{P}^1 \times \{0, \infty\}, \quad \mathrm{Gr}^{M_2} \cap \mathrm{Gr}_{\lambda^\vee} \simeq \{0, \infty\} \times \mathbb{P}^1.$$

Recall that, with the notations of Section 1.4, the fiber of $\mathcal{L}_G^{-1}$ at $e_{\lambda^\vee}$ is identified with $\mathbb{C}t^{\lambda^\vee}w_0$. Recall also that the extended affine Weyl group $W \ltimes X^\vee$ acts on the lattice $\mathrm{Hom}\,(T \times \mathbb{G}_m, \mathbb{G}_m)$ in such a way that $\lambda^\vee \cdot \tilde{\omega}_0 = \lambda + \tilde{\omega}_0$ for all $\lambda^\vee \in X^\vee$ (see [PS, Proposition 4.9.5] for instance). Thus, for any dominant coweight $\lambda^\vee$ the restriction of $\mathcal{L}_G$ to the G-orbit $G \cdot e_{\lambda^\vee}$ is the line bundle $\mathcal{L}(\lambda)$ on $G/G_{\lambda^\vee}$. In particular the restriction of the line bundle $\mathcal{L}_i$ to $\mathrm{Gr}^{M_i}_{\lambda^\vee}$ is $O_{\mathbb{P}^1}(1)$. Thus $\mathbf{e}_1 = l \otimes id$ and $\mathbf{e}_2 = id \otimes l$, where l is the product by the 1-st Chern class of $O_{\mathbb{P}^1}(1)$. The relation is obviously satisfied.

3.3. Assume that $\lambda^\vee$ is a quasi-minuscule dominant coweight. Observe that if G is of type $A_1 \times A_1$, A_2 or B_2, then the set of minuscule coweights is non empty. Thus, from Section 2.3 we can assume that G is of type G_2. Let $\alpha_1^\vee$ be the long simple coroot, and let $\alpha_2^\vee$ be the short one. Then

$$\lambda^\vee = \alpha_1^\vee + 2\alpha_2^\vee, \qquad (\alpha_2, \alpha_1^\vee) = -3, \qquad (\alpha_1, \alpha_2^\vee) = -1.$$

Set $\mathcal{L} = \mathcal{L}(\lambda)$ and $\bar{\mathcal{L}} = \mathcal{L} \cup \{e_0\}$. Then $\overline{\mathrm{Gr}}_{\lambda^\vee} \cap \mathrm{Gr}^{M_i}$ is the fixpoints set of Z_i on $\bar{\mathcal{L}}$, i.e.

$$\overline{\mathrm{Gr}}_{\lambda^\vee} \cap \mathrm{Gr}^{M_i} = \{e_0\} \cup \bigcup_{\mu^\vee \in W \cdot \lambda^\vee} \mathrm{Gr}^{M_i}_{\mu^\vee} \quad \text{where} \quad \mathrm{Gr}^{M_i}_{\mu^\vee} = {}^{Z_i}\mathcal{L}|_{M_i e_{\mu^\vee}}.$$

Assume that $\mu^\vee = w \cdot \lambda^\vee$ with $w \in W$.

(a) If $(\alpha_i, \mu^\vee) = 0$ then $\mathrm{Gr}^{M_i}_{\mu^\vee} = {}^{Z_i}\mathcal{L}|_{e_{\mu^\vee}}$. The torus T acts on the fiber $\mathcal{L}|_{e_{\mu^\vee}}$ by the character μ. Since $\mu^\vee \neq 0$ and $(\alpha_i, \mu^\vee) = 0$, necessarily $\mu(Z_i)$ is non-trivial. Thus, $\mathrm{Gr}^{M_i}_{\mu^\vee} = e_{\mu^\vee}$.

(b) If $(\alpha_i, \mu^\vee) = 2$ then $\mu^\vee = \alpha_i^\vee$ and $\mathrm{Gr}^{M_i}_{\mu^\vee} = \mathcal{L}|_{M_i e_{\alpha_i^\vee}}$. Moreover, since $\lambda^\vee$ is short and $\alpha_1^\vee$ is long we have $i = 2$.

(c) If $(\alpha_i, \mu^\vee) = 1$ then $\mathrm{Gr}_{\mu^\vee}^{M_i} = M_i e_{\mu^\vee}$ because

$$\big(\mu(Z_i) = 1 \quad \text{and} \quad \mu^\vee \in R^\vee\big) \Rightarrow \mu^\vee \in \mathbb{Z}\alpha_i^\vee \Rightarrow (\alpha_i, \mu^\vee) \neq 1.$$

In Case (b) we get $(i = 2)$

$$S_{\alpha_i^\vee}^{M_i} \cap \overline{\mathrm{Gr}}_{\lambda^\vee} = \mathcal{L}|_{U_i e_{\alpha_i^\vee}}, \quad S_{-\alpha_i^\vee}^{M_i} \cap \overline{\mathrm{Gr}}_{\lambda^\vee} = e_{-\alpha_i^\vee}, \quad S_0^{M_i} \cap \overline{\mathrm{Gr}}_{\lambda^\vee} = \bar{\mathcal{L}}^\times|_{e_{-\alpha_i^\vee}},$$

$$\text{and} \quad \overline{\mathrm{Gr}}_{\alpha_i^\vee}^{M_i} = \bar{\mathcal{L}}|_{M_i e_{\alpha_i^\vee}},$$

where the upperscript $\times$ means than the zero section has been removed. In Case (c) we get

$$S_{\mu^\vee}^{M_i} \cap \overline{\mathrm{Gr}}_{\lambda^\vee} = U_i e_{\mu^\vee}, \quad S_{\mu^\vee - \alpha_i^\vee}^{M_i} \cap \overline{\mathrm{Gr}}_{\lambda^\vee} = e_{\mu^\vee - \alpha_i^\vee}, \quad \text{and} \quad \overline{\mathrm{Gr}}_{\mu^\vee}^{M_i} = U_i e_{\mu^\vee} \cup e_{\mu^\vee - \alpha_i^\vee}.$$

Thus, for any $\mu^\vee \in X^\vee$, Claim (2.2.a) and Lemma 2.2 imply that

(d) if $\mathbf{e}_1(H_c^*(S_{\mu^\vee}, \mathcal{IC}_{\lambda^\vee})) \neq \{0\}$ then $(\alpha_1, \mu^\vee) = -1$, or $\mu^\vee = 0$, or $\mu^\vee = -\alpha_1^\vee$,

(e) if $\mathbf{f}_2(H_c^*(S_{\mu^\vee}, \mathcal{IC}_{\lambda^\vee})) \neq \{0\}$ then $(\alpha_2, \mu^\vee) = 1$, or $\mu^\vee = 0$, or $\mu^\vee = \alpha_2^\vee$.

Observe that in Case (d) the identity (2.4.c) and Lemma 2.2 imply indeed that $\mu^\vee \neq 0, -\alpha_1^\vee$, because

$$H_c^*(S_{\alpha_1^\vee}, \mathcal{IC}_{\lambda^\vee}) = H_c^*(S_{-\alpha_1^\vee}, \mathcal{IC}_{\lambda^\vee}) = \{0\}.$$

Thus, if $\mathbf{f}_2\mathbf{e}_1(H_c^*(S_{\mu^\vee}, \mathcal{IC}_{\lambda^\vee})) \neq \{0\}$ then $(\alpha_1, \mu^\vee) = -1$ and $(\alpha_2, \mu^\vee + \alpha_1^\vee) = 1$. We get $(\alpha_2, \mu^\vee) = 4$. This is not possible since $\mu^\vee \in \Omega(\lambda^\vee)$ and $\lambda^\vee$ is quasi-minuscule. Similarly, if $\mathbf{e}_1\mathbf{f}_2(H_c^*(S_{\mu^\vee}, \mathcal{IC}_{\lambda^\vee})) \neq 0$ then $(\alpha_1, \mu^\vee) = -2$. This is not possible either. Thus, the relation $[\mathbf{e}_1, \mathbf{f}_2] = 0$ is obviously satisfied. The relation $[\mathbf{e}_2, \mathbf{f}_1] = 0$ is proved in the same way.

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